Closure of the stellar hydrodynamic equations for Gaussian and ellipsoidal velocity distributions

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Abstract

The closure conditions, which make a finite set of moment equations equivalent to the collisionless Boltzmann equation, are investigated for Gaussian and ellipsoidal velocity distributions working from the complete mathematical expression for the n^{th}-order stellar hydrodynamic equation, which was explicitly obtained depending on the comoving moments in a previous paper. First, for a Schwarzschild distribution, it is proved that the whole set of hydrodynamic equations is reduced to the equations of orders $n = 0, 1, 2, 3$, owing to the recurrent form of the central moments. Furthermore, the equations of order $n = 2$ and $n = 3$ become closure conditions for higher even- and odd-order equations, respectively. An arbitrary quadratic function in the peculiar velocities, the generalised Schwarzschild distribution, is also investigated. Analogous closure conditions could be obtained from a similar recurrence law for central moments, but an alternative procedure is preferred, which consists in to expand a generalised ellipsoidal function as a power series of Schwarzschild distributions with the same mean. Then, due to the linear nature of the problem, the equivalence between the moment equations and the system of equations that Chandrasekhar had obtained working from the collisionless Boltzmann equation is borne out.

KEY WORDS: hydrodynamics – methods: analytical – stars: kinematics – galaxies: kinematics and dynamics – galaxies: statistics.

1991 MATHEMATICS SUBJECT CLASSIFICATION: 60, 62, 85.

1 Introduction

As it was suggested in a case example, the exact n^{th}-order equation derived in Cubarsi (2007), hereafter Paper I, is taken as the starting point to study the closure conditions, which make a finite set of moment equations equivalent to the collisionless Boltzmann equation. Two general cases of Gaussian and ellipsoidal trivariate velocity distributions, according to the general form of Chandrasekhar's generalised Schwarzschild functions, are used to investigate the closure problem. The analysis shows that the set of hydrodynamic equations of orders $n = 0, 1, 2, 3$ and the equations obtained by Chandrasekhar (1942) for such a type of distributions generate the same dynamical model, since the higher-order equations are found to be redundant. This is first proved for Schwarzschild distributions, which are then taken as a basis to expand a generalised ellipsoidal function as a power series of them, so that the model is extended in a natural way to the family of quadratic functions in the peculiar velocities.

For an appropriate framework of the problem let us remember that a consequence of the collisionless Boltzmann equation is that if $I_1, I_2, ..., I_6$ are any six independent isolating integrals of the equation of motion of a star for a given potential function per unit mass, $\mathcal{U}$, then the phase space density function must be of the form

$$f(t, \mathbf{r}, \mathbf{V}) = f(I_1, I_2, ..., I_6) \tag{1}$$

where the quantity on the right-hand side stands for an arbitrary function of the specified arguments, on condition that the mass of the system be finite and that the density in the phase space be non-negative. This property is an alternative form of the Liouville theorem. However, isolating integrals like the energy integral, the integral for the axial component of the angular momentum, and sometimes a third integral, are only found for all orbits under steady-state, axisymmetric potentials, or other particular potentials. In order to avoid these limitations, a functional approach may be adopted, which takes advantage of some kinematic knowledge about the stellar system. Thus, after a transient period, the velocity distribution of some stellar groups tends to be of Maxwell type, Schwarzschild type, or, in a more general way, it is ellipsoidal shaped (e.g. de Zeeuw & Lynden-Bell 1985). The functional approach then focuses in the study of one stellar population alone, which is associated with the given velocity distribution. Notice that under such a viewpoint there is no need of the collisions term in the Boltzmann equation. On the contrary, it is assumed that there are sufficient collisions to keep the system in statistical equilibrium accordingly to the specific phase space density function. Such an approach was firstly explored by Eddington (1921) and Oort (1928), and was formulated in a more general way by Chandrasekhar (1942).

Thus, for example, if the phase space density function is taken of Schwarzschild type, as Gilmore et al. (1989) point out, even though it is a quite simple case, the distribution then leads to a solution that predicts many details of the Galactic structure and kinematics, and it is possible that a realistic model could be build up as a superposition of such solutions (e.g. Cubarsi 1990), and it also leads in a natural way to Stäckel potentials and the quadratic third integral that goes with them. In order to allow some more degrees of freedom to the velocity distribution, as well as to the dynamical model, an arbitrary ellipsoidal function in the peculiar velocities was investigated by Chandrasekhar (1942). Under such an approach the dynamical model can be derived from a finite set of equations by substitution of the phase space density function into the collisionless Boltzmann equation. But we may wonder about how is it related to the moment approach and, in particular, to the infinite hierarchy of hydrodynamic equations. As it was said in Paper I, although we know that all the hydrodynamic equations must be formally fulfilled, we may guess that there is a finite subset of hydrodynamic equations which are strictly equivalent to the collisionless Boltzmann equation. Then, which are the orders of these equations? Why and which are the redundant equations? Can we explicitly write the conditions that make them redundant? All of these questions, which are specific of the velocity distribution or the integral model that have been assumed, make up the closure problem.

An interesting example of closure problem was studied by Cuddeford & Amendt (1991). Since a Schwarzschild velocity distribution may satisfy the collisionless Boltzmann equation, they adopted some closure assumptions involving the moments of the velocity distribution, which did match some known constraints between the moments of the Schwarzschild distribution, like those related to the skewness and the kurtosis of the distribution in specific directions. By this way they made the stellar hydrodynamic equations equivalent to the collisionless Boltzmann equation, as well as they obtained a phase space density function which was more general than of Schwarzschild type. However, the closure conditions they found, working even up to eighth-order moments, were only valid in a steady-state, cool and axisymmetric stellar system, with vanishing radial mean velocity. Nevertheless, most of those assumptions are not already valid in the solar neighbourhood (Alcobé & Cubarsi 2005, Cubarsi & Alcobé 2004, 2006) as it can be shown working from *HIPPARCOS* catalogue (ESA 1997). Therefore, in regard to actual data, the exact general expression of moment equations must be used to establish more general closure assumptions.

In the current work, the closure problem for Gaussian and ellipsoidal velocity distributions is analysed from a completely general approach, without any additional hypotheses, so that it may be the basis to future works on some more general distribution functions, like those obtained as finite mixtures of them. Instead of studying at first hand the generalised Schwarzschild distribution, it is better to take advantage of the algebraic simplicity of the Gaussian distribution, and in a further step, to generalise the derived results to an arbitrary ellipsoidal velocity distribution.

The plan of the paper is as follows. In the following section the basic properties of the Schwarzschild distribution and the recurrence relations of its moments are examined. In §3, after reviewing the general expression of moment equations obtained in Paper I, the closure problem for a Schwarzschild distribution is investigated. A first subsection is devoted to even-order equations for $n \geq 2$, and another subsection treats the odd-order equations for $n \geq 3$. In §4, since the whole set of hydrodynamic equations have been reduced to orders $n = 0, 1, 2, 3$, they are written in terms of $\mathbf{A}_2$ and σ, so that it is then proved that Chandrasekhar's equations can be derived from the moment equations alone. The mathematical details are shown in Appendix A. In §5 the results are extended to generalised Schwarzschild distributions, with the help of some mathematical properties contained in Appendix B. Finally, in §6, some concluding remarks are presented.

2 Schwarzschild distribution

Let us assume the phase space density function $f(t, \mathbf{r}, \mathbf{V})$ being of Schwarzschild type. According to the notation of Paper I, the stellar density was represented by

$$N(t, \mathbf{r}) = \int_{\mathbf{V}} f(t, \mathbf{r}, \mathbf{V}) \, d\mathbf{V} \tag{2}$$

and the stellar mean velocity, or velocity of the centroid, was given by

$$\mathbf{v}(t, \mathbf{r}) = \frac{1}{N(t, \mathbf{r})} \int_{\mathbf{V}} \mathbf{V} \, f(t, \mathbf{r}, \mathbf{V}) \, d\mathbf{V} \tag{3}$$

Thus, if the peculiar velocity of a star is denoted by $\mathbf{u} = \mathbf{V} - \mathbf{v}(t, \mathbf{r})$, the symmetric tensor of the n^{th}-order central moments can be obtained from the expected value

$$\mathbf{M}_n(t, \mathbf{r}) = E[(\mathbf{u})^n] = \frac{1}{N(t, \mathbf{r})} \int_{\mathbf{V}} (\mathbf{V} - \mathbf{v}(t, \mathbf{r}))^n f(t, \mathbf{r}, \mathbf{V})\, d\mathbf{V}, \quad n \geq 0 \tag{4}$$

where $(\cdot)^n$ stands for the n^{th}-tensor power, and the tensor of pressures is defined as $\mathbf{P}_n = N\,\mathbf{M}_n$. Obviously, $\mathbf{M}_0 = 1$, $\mathbf{M}_1 = \mathbf{0}$.

Then, the phase space density function f corresponding to a Schwarzschild distribution can be written in the form $\phi(t, \mathbf{r}, \mathbf{u}) = f(t, \mathbf{r}, \mathbf{u} + \mathbf{v}(t, \mathbf{r}))$ according to

$$\phi(t, \mathbf{r}, \mathbf{u}) = e^{-\frac{1}{2}(Q+\sigma)}, \quad Q = \mathbf{u}^{\mathrm{T}} \cdot \mathbf{A}_2 \cdot \mathbf{u} \tag{5}$$

where Q is a quadratic, positive definite form, with $\mathbf{A}_2(t, \mathbf{r})$ a second-rank symmetric tensor and $\sigma(t, \mathbf{r})$ a scalar function, which are continuous and differentiable in both arguments. Hence, the distribution is of Gaussian type in the peculiar velocities, although it can be multiplied by an arbitrary function of time and position. In such a way the quadratic form Q can give account of the three aforesaid isolating integrals of star motions, so that, in general, it is allowing some friction phenomena which are quantified by the off-diagonal second central moments of the distribution. As it is known, the second moments of the Schwarzschild distribution satisfy

$$\mathbf{M}_2 = \mathbf{A}_2^{-1} \tag{6}$$

and all the odd-order central moments are obviously null. Let us remark that the velocity moments do not depend on the function σ appearing in Eq. 5. Only the stellar density, which is obtained from Eq. 2, depends on it according to

$$N(t, \mathbf{r}) = (2\pi)^{\frac{3}{2}}\, |A|^{-\frac{1}{2}} e^{-\frac{1}{2}\sigma}, \quad |A| = \det \mathbf{A}_2 \tag{7}$$

The more general way to characterise such a trivariate distribution is from its cumulants, which, in addition and opposite to the central moments, have unbiased sample estimators. In general, the relationship between moments with arbitrary mean $\mathbf{M}_n$ and cumulants $\mathbf{K}_n$ (Stuart & Ord 1987, §13.11-15) is given by

$$\begin{array}{l} \mathbf{M}_1 = \mathbf{K}_1 \\ \mathbf{M}_2 = \mathbf{K}_2 + \mathcal{S}(\mathbf{K}_1 \otimes \mathbf{K}_1) \\ \mathbf{M}_3 = \mathbf{K}_3 + \mathcal{S}(\mathbf{K}_2 \otimes \mathbf{K}_1) + \mathcal{S}(\mathbf{K}_1 \otimes \mathbf{K}_1 \otimes \mathbf{K}_1) \\ \mathbf{M}_4 = \mathbf{K}_4 + \mathcal{S}(\mathbf{K}_3 \otimes \mathbf{K}_1) + \mathcal{S}(\mathbf{K}_2 \otimes \mathbf{K}_2) + \mathcal{S}(\mathbf{K}_2 \otimes \mathbf{K}_1 \otimes \mathbf{K}_1) + \mathcal{S}(\mathbf{K}_1 \otimes \mathbf{K}_1 \otimes \mathbf{K}_1 \otimes \mathbf{K}_1) \\ \vdots \end{array} \tag{8}$$

where the notation for symmetrised tensors defined in §4 is applied. If centered variables are used, then the odd-order cumulants vanish and the Gaussian distribution remains characterised only from its second cumulants $\mathbf{K}_2 = \mathbf{M}_2$. In other words, the symmetry properties of the Gaussian distribution do also provide vanishing even-order cumulants $\mathbf{K}_n = 0$ for $n \geq 4$ (Stuart & Ord 1987, §15.3). Then, under those premises, the relationships of Eq. 8 are reduced to the following ones, which are written by using the star product notation also defined in §4,

$$
\begin{aligned}
&\mathbf{M}_4 = \mathcal{S}(\mathbf{M}_2 \otimes \mathbf{M}_2) = 3\,\mathbf{M}_2 \star \mathbf{M}_2 \\
&\mathbf{M}_6 = \mathcal{S}(\mathbf{M}_2 \otimes \mathbf{M}_2 \otimes \mathbf{M}_2) = 15\,\mathbf{M}_2 \star \mathbf{M}_2 \star \mathbf{M}_2 \\
&\vdots \\
&\mathbf{M}_{2n} = \mathcal{S}(\textstyle\bigotimes^n \mathbf{M}_2) = C_n \, \overbrace{\mathbf{M}_2 \star \cdots \star \mathbf{M}_2}^{n}, \quad C_n = \dfrac{(2n)!}{2^n n!} \\
&\vdots
\end{aligned} \tag{9}
$$

Therefore, we can easily obtain the relationship between two consecutive even-order moments. The previous coefficient C_n satisfies the recurrence relation

$$
C_{n+1} = \frac{(2n+2)(2n+1)!(2n)!}{2^n 2(n+1)n!} = (2n+1)\, C_n \tag{10}
$$

which allow us to write Eq. 9 as

$$
\mathbf{M}_{2n+2} = \tfrac{C_{n+1}}{C_n}\, \mathcal{S}(\textstyle\bigotimes^n \mathbf{M}_2) \star \mathbf{M}_2 = (2n+1)\, \mathbf{M}_{2n} \star \mathbf{M}_2 \tag{11}
$$

The relation Eq. 11 of moments recurrence will be used to simplify and to reduce higher-order moment equations to lower-order ones, so that such a relationship will provide the key to the closure problem.

3 Closure problem

In Paper I the general expression for an arbitrary n^{th}-order hydrodynamic equation, written in terms of the generalised tensor of pressures $\mathbf{P}_n$, was obtained as[1]

$$
\frac{\partial \mathbf{P}_n}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{P}_n + n \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{v} + \nabla_{\mathbf{r}} \mathcal{U} \right) \star \mathbf{P}_{n-1} + \nabla_{\mathbf{r}} \cdot \mathbf{P}_{n+1} + n \,(\mathbf{P}_n \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} + (\nabla_{\mathbf{r}} \cdot \mathbf{v})\, \mathbf{P}_n = (\mathbf{0})^n \tag{12}
$$

Such a conservation law for pressures can be written in terms of the comoving moments as follows. For $n = 0$ the continuity equation yields

$$
\frac{\partial N}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} N + N \nabla_{\mathbf{r}} \cdot \mathbf{v} = 0 \tag{13}
$$

[1] Let us remember the notation used for symmetrised tensor products (Cubarsi 1992), which worthy simplifies the equations. In general, if $\mathbf{A}_m$ and $\mathbf{B}_n$ are two m- and n-rank symmetric tensors, we can define the tensor $\mathbf{A}_m \star \mathbf{B}_n$ as the obtained by symmetrising the tensor product $\mathbf{A}_m \otimes \mathbf{B}_n$, and by normalising then with respect to the number of summation terms, T. The result is a $(m+n)$-rank symmetric tensor, whose components are

$$
(\mathbf{A}_m \star \mathbf{B}_n)_{i_1 i_2 \dots i_{m+n}} = \frac{1}{T}\, \mathcal{S}\,(\mathbf{A}_m \otimes \mathbf{B}_n)_{i_1 i_2 \dots i_{m+n}} = \frac{1}{T} \sum_{\substack{\alpha i_1 < \ldots < \alpha i_m \\ \alpha i_{m+1} < \ldots < \alpha i_{m+n}}} A_{\alpha i_1 \dots \alpha i_m} B_{\alpha i_{m+1} \dots \alpha i_{m+n}}
$$

where α belongs to the symmetric group $\mathrm{S}(m+n)$. If both tensors are different ones, then $T = \frac{(m+n)!}{n!m!}$. Notice that, in particular, if $\mathbf{A}_m = \mathbf{B}_n$ the number of summation terms is $T = \dfrac{(2n)!}{2!n!n!}$, and for the symmetric tensor product $\mathcal{S}(\bigotimes^k \mathbf{A}_n)$ the number of terms is $T = \dfrac{(kn)!}{k!(n!)^k}$.

For $n = 1$ the momentum equation, also known as Jeans equation, can be expressed as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{v} + \nabla_{\mathbf{r}} \mathcal{U} = -(\nabla_{\mathbf{r}} \ln N + \nabla_{\mathbf{r}}) \cdot \mathbf{M}_2 \tag{14}$$

and, in general, Eq. 14 may be introduced into the higher-order equations to replace the terms depending on the potential function, so that they remain written in terms of the comoving moments, for $n \geq 2$, in the form

$$\frac{\partial \mathbf{M}_n}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{M}_n - n \; [(\nabla_{\mathbf{r}} \ln N + \nabla_{\mathbf{r}}) \cdot \mathbf{M}_2] \star \mathbf{M}_{n-1} + (\nabla_{\mathbf{r}} \ln N + \nabla_{\mathbf{r}}) \cdot \mathbf{M}_{n+1} + n \; (\mathbf{M}_n \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} = (\mathbf{0})^n \tag{15}$$

Let us note that the equations Eq. 13 and Eq. 14, for $n = 0$ and $n = 1$, contain four different scalar equations, which involve a set of eleven unknown scalar functions, namely $N, \mathcal{U}, \mathbf{v}$ and the symmetric tensor $\mathbf{M}_2$. Even in the case of taking into account higher-order equations the system remains always open, since by picking up the m^{th}-equation, which contains $\binom{m+2}{2}$ scalar equations, we are also introducing as many as $\binom{m+3}{2}$ new unknowns, which are the different components of the tensor $\mathbf{M}_{m+1}$. Thus, the current approach to the closure problem will consist in to investigate how the velocity distribution function, Eq. 5, and the recurrent moment relations it provides, Eq. 11, allow to reduce the infinity of equations involved in Eq. 15 to any finite subset of them.

3.1 Even-order equations, $n \geq 2$

For even-order equations, $n = 2k$ and $k \geq 1$, bearing in mind that the odd-moments are null, Eq. 15 becomes

$$\frac{\partial \mathbf{M}_{2k}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{M}_{2k} + 2k \; (\mathbf{M}_{2k} \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} = (\mathbf{0})^{2k} \tag{16}$$

which, by substitution of the moment expression Eq. 9, is transformed into

$$\tfrac{\partial}{\partial t} \mathcal{S}(\otimes^k \mathbf{M}_2) + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathcal{S}(\otimes^k \mathbf{M}_2) + 2k \, (\mathcal{S}(\otimes^n \mathbf{M}_2) \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} = (\mathbf{0})^{2k} \tag{17}$$

After some algebra, we have

$$\begin{gathered} \tfrac{C_k}{C_{k-1}} \, k \, \mathcal{S}(\otimes^{k-1} \mathbf{M}_2) \star \tfrac{\partial \mathbf{M}_2}{\partial t} + \tfrac{C_k}{C_{k-1}} \, k \, \mathcal{S}(\otimes^{k-1} \mathbf{M}_2) \star \, (\mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{M}_2) + \\ + \tfrac{C_k}{C_{k-1}} \, 2k \, \mathcal{S}(\otimes^{k-1} \mathbf{M}_2) \star (\mathbf{M}_2 \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} = (\mathbf{0})^{2k} \end{gathered} \tag{18}$$

And, by taking into account Eq. 11 we can write

$$\frac{C_k}{C_{k-1}} \mathbf{M}_{2k-2} \star \left[\frac{\partial \mathbf{M}_2}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{M}_2 + 2 \, (\mathbf{M}_2 \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} \right] = (\mathbf{0})^{2k} \tag{19}$$

Since $\mathbf{M}_{2k-2}$ never vanishes, all the even-order equations, $n \geq 2$, are then reduced to the moment equation of second-order,

$$\frac{\partial \mathbf{M}_2}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{M}_2 + 2 \, (\mathbf{M}_2 \cdot \nabla_{\mathbf{r}}) \star \mathbf{v} = (\mathbf{0})^2 \tag{20}$$

Therefore, such a relationship, along with the moments recurrence given by Eq. 11, provides a closure condition for the even-order hydrodynamic equations.

3.2 Odd-order equations, $n \geq 3$

In a similar way, for odd-order equations, $n = 2k+1$, $k \geq 1$, and provided that the odd-order moments are null, from Eq. 15 we can write

$$-(2k+1)\ \left[(\nabla_{\mathbf{r}} \ln N + \nabla_{\mathbf{r}}) \cdot \mathbf{M}_2)\right] \star \mathbf{M}_{2k} + (\nabla_{\mathbf{r}} \ln N + \nabla_{\mathbf{r}}) \cdot \mathbf{M}_{2k+2} = (\mathbf{0})^{2k+1} \tag{21}$$

Then, by substitution of the recurrence law for the moments, Eq. 11, into the foregoing equation we have

$$\begin{gathered} -(2k+1)\ (\nabla_{\mathbf{r}} \ln N \cdot \mathbf{M}_2) \star \mathbf{M}_{2k} - (2k+1)\ (\nabla_{\mathbf{r}} \cdot \mathbf{M}_2) \star \mathbf{M}_{2k} + \\ +(2k+1)\, \nabla_{\mathbf{r}} \ln N \cdot (\mathbf{M}_2 \star \mathbf{M}_{2k}) + (2k+1)\, \nabla_{\mathbf{r}} \cdot (\mathbf{M}_2 \star \mathbf{M}_{2k}) = (\mathbf{0})^{2k+1} \end{gathered} \tag{22}$$

In order to simplify the previous equation, by regarding the dependence of $\mathbf{M}_{2k}$ in terms of $\mathbf{M}_2$ given by Eq. 9, and since $\nabla_{\mathbf{r}} \ln N$ is a vector, we use the equivalence[2]

$$(\nabla_{\mathbf{r}} \ln N \cdot \mathbf{M}_2) \star \mathbf{M}_{2k} = \nabla_{\mathbf{r}} \ln N \cdot (\mathbf{M}_2 \star \mathbf{M}_{2k}) \tag{23}$$

so that Eq. 22 yields

$$-(2k+1)\ (\nabla_{\mathbf{r}} \cdot \mathbf{M}_2) \star \mathbf{M}_{2k} + (2k+1)\, \nabla_{\mathbf{r}} \cdot (\mathbf{M}_2 \star \mathbf{M}_{2k}) = (\mathbf{0})^{2k+1} \tag{24}$$

And now, to further simplify Eq. 24, and once more by taking into account Eq. 9, we use the following identity[3]

$$\nabla_{\mathbf{r}} \cdot (\mathbf{M}_2 \star \mathbf{M}_{2k}) = (\nabla_{\mathbf{r}} \cdot \mathbf{M}_2) \star \mathbf{M}_{2k} + k\, (\mathbf{M}_2 \cdot \nabla_{\mathbf{r}}) \star \mathbf{M}_{2k} \tag{25}$$

so that Eq. 24 takes the form

$$(\mathbf{M}_2 \cdot \nabla_{\mathbf{r}}) \star \mathbf{M}_{2k} = (\mathbf{0})^{2k+1} \tag{26}$$

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[2] In general, for a vector $\mathbf{a}$ and for any symmetric tensors $\mathbf{A}_m$ and $\mathbf{B}_n$, the following equivalence is satisfied,

$$\mathbf{a} \cdot \mathcal{S}(\mathbf{A}_m \otimes \mathbf{B}_n) = \mathcal{S}((\mathbf{a} \cdot \mathbf{A}_m) \otimes \mathbf{B}_n) + \mathcal{S}(\mathbf{A}_m \otimes (\mathbf{a} \cdot \mathbf{B}_n))$$

[3] In general, for any symmetric tensors $\mathbf{A}_m$ and $\mathbf{B}_n$, the following equality is satisfied,

$$\nabla \cdot \mathcal{S}(\mathbf{A}_m \otimes \mathbf{B}_n) = \mathcal{S}((\nabla \cdot \mathbf{A}_m) \otimes \mathbf{B}_n) + \mathcal{S}((\mathbf{A}_m \cdot \nabla) \otimes \mathbf{B}_n) + \mathcal{S}((\nabla \otimes \mathbf{A}_m) \cdot \mathbf{B}_n) + \mathcal{S}(\mathbf{A}_m \otimes (\nabla \cdot \mathbf{B}_n))$$

in particular, if $\mathbf{A}_m = \mathbf{B}_n$, we have

$$\nabla \cdot \mathcal{S}(\mathbf{A}_m \otimes \mathbf{A}_m) = \mathcal{S}((\nabla \cdot \mathbf{A}_m) \otimes \mathbf{A}_m) + \mathcal{S}((\mathbf{A}_m \cdot \nabla) \otimes \mathbf{A}_m)$$

and, if $\mathbf{B}_n = \overbrace{\mathbf{A}_m \star \cdots \star \mathbf{A}_m}^{k}$, the equality yields

$$\nabla \cdot \mathcal{S}(\mathbf{A}_m \otimes \mathbf{B}_n) = \mathcal{S}((\nabla \cdot \mathbf{A}_m) \otimes \mathbf{B}_n) + k\, \mathcal{S}((\mathbf{A}_m \cdot \nabla) \otimes \mathbf{B}_n)$$

Finally, since $\mathbf{M}_{2k} = (2k-1)\,\mathbf{M}_2 \star \mathbf{M}_{2k-2}$, and $\mathbf{M}_{2k-2}$ is always non-null for $k \geq 1$, Eq. 26 is reduced to the third-order equation

$$(\mathbf{M}_2 \cdot \nabla_{\mathbf{r}}) \star \mathbf{M}_2 = (\mathbf{0})^3 \tag{27}$$

Thus, the foregoing expression stands for all the odd-order equations, $n = 2k+1 \geq 3$, and, in virtue of the moments recurrence Eq. 11, such a relation provides a closure condition for odd-order hydrodynamic equations in terms of the second central moments $\mathbf{M}_2$.
In conclusion, for a velocity distribution of Schwarzschild type, if the closure conditions given by Eq. 11, Eq. 20 and Eq. 27 are satisfied, then all the moment equations are reduced to the four equations of orders $n = 0, 1, 2, 3$, which are a set of twenty scalar equations.

4 Equations for $\mathbf{A}_2$ and σ

In the previous sections we have been left only with four independent hydrodynamic equations involving the statistics N, $\mathbf{v}$ and $\mathbf{M}_2$. The equations for conservation of mass and momentum, $n = 0, 1$, do not provide the same model as the collisionless Boltzmann equation, but, since the higher even-order moments can be expressed in the recurrence form of Eq. 11, and, in virtue of the equations of orders $n = 2$ and $n = 3$ that are acting as closure conditions, then the four hydrodynamic equations are completely equivalent to the Boltzmann equation.
In the current section two implications of such a result are proved. Firstly, the referred set of equations is totally equivalent to the system of equations obtained by Chandrasekhar (1942) for generalised Schwarzschild distributions and, secondly, our result for Gaussian velocity distributions is also valid for generalised ellipsoidal velocity distributions. The reason of proceeding in two steps is for mathematical simplicity. It is thus formally proved not only that Chandrasekhar's equations are equivalent to a subset of hydrodynamic equations (de Orús 1952, Juan-Zornoza 1995) but also that, because of the closure conditions, they are equivalent to the infinite hierarchy of hydrodynamic equations. Indeed, de Orús (1952) had proved that if Chandrasekhar's equations were fulfilled, then the continuity equation and Jean's equation were also satisfied. On the other hand, working from velocity moments up to fourth-order, Juan-Zornoza (1995) showed that Chandrasekhar's equations could be derived from the first four hydrodynamic equations. Now, from a new and more general approach, the whole set of hydrodynamic equations and velocity moments have been taken into account.
Therefore, since Chandrasekhar's equations provide the functional dependence of $\mathbf{A}_2$ and σ, let us transform the moment equations in terms of those quantities. In the Appendix A.1 we find the algebraic details showing that the Eq. 27, which corresponds to the equation of order $n = 3$, and only involves the tensor of moments $\mathbf{M}_2$, is equivalent to the following condition on the tensor $\mathbf{A}_2$,

$$3\,\nabla_{\mathbf{r}} \star \mathbf{A}_2 = (\mathbf{0})^3 \tag{28}$$

From Eq. 27, in the Appendix A.2 it is also derived an auxiliary property which will be used later.
The Eq. 28 represents a set of 10 scalar, first-order linear equations in partial derivatives for the elements of the symmetric tensor $\mathbf{A}_2$.
Working from the hydrodynamic equation of order $n = 2$, Eq. 20, along with Eq. 28, in the Appendix A.3 the following relationship is obtained,

$$\frac{\partial \mathbf{A}_2}{\partial t} - 2\,\nabla_{\mathbf{r}} \star (\mathbf{A}_2 \cdot \mathbf{v}) = (\mathbf{0})^2 \tag{29}$$

which stands for a set of 6 scalar first-order linear partial differential equations for $\mathbf{A}_2$ and $\mathbf{v}$. Also, as an auxiliary property, the divergence of the mean velocity is determined in the Appendix A.4.
By using both aforesaid auxiliary properties, in the Appendix A.5 it is obtained the relationship which is equivalent to the hydrodynamic equation of order $n = 1$, Eq. 14,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \mathbf{v} + \nabla_{\mathbf{r}} \mathcal{U} = -\frac{1}{2} \mathbf{A}_2^{-1} \cdot \nabla_{\mathbf{r}} \sigma \tag{30}$$

whose components are 3 scalar equations, the only ones which involve the potential function.
Notice that if the phase space density function is strictly of Gaussian type, with $\sigma(t, \mathbf{r}) = 0$, then the centroid motion does not change neither due to pressure nor to viscosity. In other words, there are no transport phenomena and the centroid moves like a particle under the gravitational potential.
Finally, in the Appendix A.6, the continuity equation Eq. 13, for order $n = 0$, is proved equivalent to the following condition

$$\frac{\partial \sigma}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \sigma = 0 \tag{31}$$

which is one scalar linear differential equation for σ, giving account of the conservation of such a quantity along the centroid path.

5 Generalised Schwarzschild distribution

In this subsection it is shown that if the moment equations are fulfilled for a Schwarzschild distribution $e^{-\frac{1}{2}(Q+\sigma)}$, then they are also satisfied for a generalised ellipsoidal distribution in the form

$$f(t, \mathbf{r}, \mathbf{V}) = \psi(Q + \sigma) \tag{32}$$

where ψ is an arbitrary function of the specified argument as defined in Eq. 5.
For such a distribution the even-order cumulants are not null, as they were in the Gaussian case, although all the even-order central moments can also be computed in terms of the second ones. Thus, the odd-order moments obviously vanish, and the even moments can be expressed from symmetrised tensor products of the second moments, similarly to the Gaussian case, but with a new factor that depends on σ (de Orús 1977) through the integral[4]

$$\phi_n(\sigma) = \int_0^{\infty} Q^{\frac{n}{2}} \psi(Q + \sigma) Q^{\frac{1}{2}} dQ \tag{33}$$

for n even or null. Then, Eq. 2 provides a stellar density in the form

$$N(t, \mathbf{r}) = 2\pi \, |A|^{-\frac{1}{2}} \phi_0(\sigma) \tag{34}$$

and, by using the notation which was introduced in Eqs. 9 and 10, the even-order moments can be written as

[4]The integral is related to the Mellin transform, which, for a function $f(x)$, is defined as $\varphi(s) = \int_0^{\infty} x^{s-1} f(x) dx$ (e.g. Ditkin & Prudnikov 1965).

$$\mathbf{M}_{2n} = \frac{1}{C_{n+1}} \frac{\phi_{2n}(\sigma)}{\phi_0(\sigma)} C_n \overbrace{\mathbf{A}_2^{-1} \star \cdots \star \mathbf{A}_2^{-1}}^{n} = \frac{1}{C_{n+1}} \frac{\phi_{2n}(\sigma)}{\phi_0(\sigma)} \mathcal{S}(\bigotimes^n \mathbf{A}_2^{-1}) \tag{35}$$

In order to investigate the closure conditions, a similar procedure as for Gaussian distributions could be applied, but in the current case it is quite more long and tedious, since the moments depend on the functions $\phi_k(\sigma)$. Hence I shall use an indirect and shorter approach.
In the Appendix B it is shown that the family of Schwarzschild functions

$$\left\{e^{-\frac{1}{2}(Q+\sigma)\,k}\right\}_k, \quad k \in \mathcal{N} - \{0\} \tag{36}$$

constitutes a non-orthogonal basis of the space of square-integrable functions over the interval $(0, +\infty)$ for the variable Q, so that the integrals of Eq. 33, which arise when computing the moments, are convergent. Then the velocity distribution can be formally expressed as a linear combination of them, according to

$$\psi(Q+\sigma) = \sum_{k=1}^{\infty} \gamma_{k-1} e^{-\frac{1}{2}(Q+\sigma)\,k} \tag{37}$$

Therefore, an arbitrary quadratic distribution $\psi(Q+\sigma)$ can be written as a convergent series of Gaussian functions in the peculiar velocities, $e^{-\frac{1}{2}(Q+\sigma)\,k}$, for $k \geq 1$, all of them having zero mean. Hence, all the centroids of the partial distributions –for each term of the series– have the same mean velocity $\mathbf{v}$. Under those premises, not only the collisionless Boltzmann equation is linear for the phase space density function, but also all the hydrodynamic equations, Eq. 12, are linear in the pressures, since the total n^{th}-pressure is simply the sum of the partial n^{th}-pressures. Notice that if the mean velocities were different for each partial distribution, as the lineality on the pressures would not then hold, the hydrodynamic equations might be considered separately for each distribution component.
Then, the tensor involved in the exponent of the k^{th}-term in Eq. 37 is $k\,\mathbf{A}_2$, and the accompanying function of t and $\mathbf{r}$ is $k\,\sigma$. In addition, we can see that for each Schwarzschild component the moment equations in terms of $\mathbf{A}_2$ and σ, Eqs. 28, 29, 30 and 31, remain invariant whether $\mathbf{A}_2$ and σ are respectively exchanged by $k\,\mathbf{A}_2$ and $k\,\sigma$. Therefore, due to the linear condition of the problem, if each Gaussian summation term in Eq. 36 satisfies the moment equations, then an arbitrary generalised ellipsoidal distribution $\psi(Q+\sigma)$, according to Eq. 37, do satisfy them too.

6 Concluding remarks

The general expression for the moment equations that was derived in Paper I has proved useful to study the closure problem for trivariate Gaussian and ellipsoidal velocity distributions, without any additional hypotheses. By this way, the infinite hierarchy of moment equations and unknowns are reduced to a finite number of them, which are equivalent to the collisionless Boltzmann equation, so that a feasible dynamical model is available.
In general, under the functional approach, the statistical properties of the velocity distribution function may be derived and used to reduce the whole set of moment equations to a finite subset. For trivariate Schwarzschild distributions a recurrence relation between even central moments has been found, which allow to reduce the moment equations only to four different orders, being the even-order equations for $n \geq 2$ equivalent to the one of order $n = 2$, and the odd-order equations for $n \geq 3$ equivalent to the one of order $n = 3$. Therefore, the equations for mass

and momentum transfer do not generate the same model as the collisionless Boltzmann equation, whereas if the moment equations of orders $n = 2$ and $n = 3$ are considered, along with the relation of moments recurrence, which are acting as closure conditions, the model they provide is the same as the one Chandrasekhar had derived by working from Boltzmann equation, as it has been proved in the Appendix A. For generalised Schwarzschild distributions, a similar recurrence law for central moments would also provide us with some closure conditions, but, for the sake of mathematical simplicity, an alternative method has been preferred. The result derived for Schwarzschild distributions has been extended in a natural way, owing to the linear nature of the problem, to generalised Schwarzschild distributions, which can be expanded as a power series of Schwarzschild functions with the same mean velocity.
Therefore, both case examples show the advantage of having derived the full form of moment equations, which may be used in future works on some more general distribution functions or analytical integral models, and on the closure conditions they provide.

Appendix A

Moment equations for a Schwarzschild distribution in terms of $\mathbf{A}_2$ and σ

A.1. Equation of order $n = 3$

In components, Eq. 6 is equivalent to

$$M_{i\alpha} A_{\alpha j} = \delta_{ij} \tag{38}$$

By taking gradient in both sides we get the identity

$$\frac{\partial}{\partial r_k} \left(M_{i\alpha} A_{\alpha j} \right) = \frac{\partial}{r_k} \left(\delta_{ij} \right) = 0 \tag{39}$$

which is valid for any index k of the derivative variable, either with or without contraction of indices, and it is also valid if the time derivative is taken.
The following relationship is then satisfied

$$\frac{\partial M_{i\alpha}}{\partial r_k} A_{\alpha j} = -M_{i\alpha} \frac{\partial A_{\alpha j}}{\partial r_k} \tag{40}$$

If the equation Eq. 27 corresponding to order $n = 3$ is contracted three times with the tensor $\mathbf{A}_2 \otimes \mathbf{A}_2$, which is always non-null, we have

$$A_{ij} A_{kl} \left(M_{i\alpha} \frac{\partial M_{jk}}{\partial r_\alpha} + M_{j\alpha} \frac{\partial M_{ik}}{\partial r_\alpha} + M_{k\alpha} \frac{\partial M_{ij}}{\partial r_\alpha} \right) = 0 \tag{41}$$

By taking into account Eq. 38, we may then write

$$\delta_{j\alpha} A_{kl} \frac{\partial M_{jk}}{\partial r_\alpha} + \delta_{i\alpha} A_{kl} \frac{\partial M_{ik}}{\partial r_\alpha} + \delta_{l\alpha} A_{ij} \frac{\partial M_{ij}}{\partial r_\alpha} = A_{kl} \frac{\partial M_{jk}}{\partial r_j} + A_{kl} \frac{\partial M_{ik}}{\partial r_i} + A_{ij} \frac{\partial M_{ij}}{\partial r_l} = 0 \tag{42}$$

and by Eq. 40, as well as by changing the sign, we get

$$M_{jk}\frac{\partial A_{kl}}{\partial r_j}+M_{ik}\frac{\partial A_{kl}}{\partial r_i}+M_{ij}\frac{\partial A_{ij}}{\partial r_l}=0 \tag{43}$$

If some repeated indices are changed, we can write

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j}+\frac{\partial A_{jk}}{\partial r_i}+\frac{\partial A_{ij}}{\partial r_k}\right)=0 \tag{44}$$

Therefore, since the double contraction of indices is carried out with the non-null symmetric tensor $\mathbf{M}_2$, inverse of $\mathbf{A}_2$, which is associated with a positive definite quadratic form, we are led to the equation

$$3\,\nabla_{\mathbf{r}}\star\mathbf{A}_2=(\mathbf{0})^3 \tag{45}$$

Such a relation gives then account of the spatial dependence of the tensor $\mathbf{A}_2$.

A.2. Property

We deduce a consequence of Eq. 44, which will be useful in the following sections. By applying the relation Eq. 40, since the tensors there involved are symmetric, the left-hand side of Eq. 44 can be written as

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j}+\frac{\partial A_{jk}}{\partial r_i}+\frac{\partial A_{ij}}{\partial r_k}\right)=2M_{ij}\frac{\partial A_{ik}}{\partial r_j}+M_{ij}\frac{\partial A_{ij}}{\partial r_k}=-2A_{ik}\frac{\partial M_{ij}}{\partial r_j}+M_{ij}\frac{\partial A_{ij}}{\partial r_k} \tag{46}$$

Now, if $\bar{A}_{ij}$ denotes the cofactor of the element A_{ij}, which is the same one as for its transposed element A_{ji}, then the relation Eq. 6 obviously implies

$$M_{ij}=\frac{\bar{A}_{ij}}{|A|} \tag{47}$$

Hence Eq. 46 can be converted into

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j}+\frac{\partial A_{jk}}{\partial r_i}+\frac{\partial A_{ij}}{\partial r_k}\right)=-2A_{ik}\frac{\partial M_{ij}}{\partial r_j}+\frac{\bar{A}_{ij}}{|A|}\frac{\partial A_{ij}}{\partial r_k} \tag{48}$$

It is well known that if the tensor $\mathbf{A}_2$ depends on a variable ξ, then the relation

$$\frac{\partial|A|}{\partial\xi}=\frac{\partial A_{ij}}{\partial\xi}\bar{A}_{ij} \tag{49}$$

is satisfied[5]. Hence, by writing $\xi=r_k$ we have

[5] The determinant of $\mathbf{A}_2$ can be expressed as $|A|=\epsilon_{i_1\ldots i_n}A_{1i_1}\cdots A_{ni_n}$, where $\epsilon_{i_1\ldots i_n}$ denotes the Levi-Civitta tensor. Then,

$$\frac{\partial|A|}{\partial\xi}=\epsilon_{i_1\ldots i_n}\left(\frac{\partial A_{1i_1}}{\partial\xi}A_{2i_2}\ldots A_{ni_n}+A_{1i_1}\frac{\partial A_{2i_2}}{\partial\xi}\ldots A_{ni_n}+\cdots+A_{1i_1}\ldots\frac{\partial A_{ni_n}}{\partial\xi}\right)=$$

$$=\frac{\partial A_{1i_1}}{\partial\xi}\bar{A}_{1i_1}+\frac{\partial A_{2i_2}}{\partial\xi}\bar{A}_{2i_2}+\cdots+\frac{\partial A_{ni_n}}{\partial\xi}\bar{A}_{ni_n}=\frac{\partial A_{ij}}{\partial\xi}\bar{A}_{ij}$$

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j}+\frac{\partial A_{jk}}{\partial r_i}+\frac{\partial A_{ij}}{\partial r_k}\right)=-2A_{ik}\frac{\partial M_{ij}}{\partial r_j}+\frac{1}{|A|}\frac{\partial |A|}{\partial r_k} \tag{50}$$

Therefore, bearing in mind that $\mathbf{M}_2=\mathbf{A}_2^{-1}$, it is fulfilled

$$3\mathbf{M}_2:(\nabla_{\mathbf{r}}\star\mathbf{A}_2)=-2\mathbf{A}_2\cdot(\nabla_{\mathbf{r}}\cdot\mathbf{M}_2)+\nabla_{\mathbf{r}}\ln|A| \tag{51}$$

Nevertheless, in virtue of Eq. 44, the left-hand side of the above equation is zero. Hence, the following relationship is satisfied,

$$\mathbf{A}_2\cdot(\nabla_{\mathbf{r}}\cdot\mathbf{M}_2)=\nabla_{\mathbf{r}}\ln|A|^{\frac{1}{2}} \tag{52}$$

which, by taking dot product with $\mathbf{M}_2$, can also be written as

$$\nabla_{\mathbf{r}}\cdot\mathbf{M}_2=\mathbf{M}_2\cdot\nabla_{\mathbf{r}}\ln|A|^{\frac{1}{2}} \tag{53}$$

A.3. Equation of order $n=2$

For the second order hydrodynamic equation, if we take the colon product of $\mathbf{A}_2\otimes\mathbf{A}_2$ with Eq. 20, we have

$$A_{ik}A_{jl}\left(\frac{\partial M_{ij}}{\partial t}+v_\alpha\frac{\partial M_{ij}}{\partial r_\alpha}+M_{i\alpha}\frac{\partial v_j}{\partial r_\alpha}+M_{j\alpha}\frac{\partial v_i}{\partial r_\alpha}\right)=0 \tag{54}$$

By taking into account Eq. 40 we can write

$$-A_{ik}M_{ij}\frac{\partial A_{jl}}{\partial t}-A_{ik}M_{ij}v_\alpha\frac{\partial A_{jl}}{\partial r_\alpha}+A_{ik}M_{i\alpha}A_{jl}\frac{\partial v_j}{\partial r_\alpha}+A_{ik}A_{jl}M_{j\alpha}\frac{\partial v_i}{\partial r_\alpha}=0 \tag{55}$$

and now, by Eq. 38, we have

$$\begin{aligned}&-\delta_{kj}\frac{\partial A_{jl}}{\partial t}-\delta_{kj}v_\alpha\frac{\partial A_{jl}}{\partial r_\alpha}+\delta_{k\alpha}A_{jl}\frac{\partial v_j}{\partial r_\alpha}+A_{ik}\delta_{j\alpha}\frac{\partial v_i}{\partial r_\alpha}=\\&=-\frac{\partial A_{kl}}{\partial t}-v_\alpha\frac{\partial A_{kl}}{\partial r_\alpha}+A_{jl}\frac{\partial v_j}{\partial r_k}+A_{ik}\frac{\partial v_i}{\partial r_j}=0\end{aligned} \tag{56}$$

On the other hand, in Eq. 45, if we take the inner product with v_α, we obtain the identity

$$-v_\alpha\frac{\partial A_{jl}}{\partial r_\alpha}=v_\alpha\frac{\partial A_{j\alpha}}{\partial r_l}+v_\alpha\frac{\partial A_{\alpha l}}{\partial r_j} \tag{57}$$

which, by substitution in Eq. 56, yields

$$-\frac{\partial A_{kl}}{\partial t}+v_\alpha\frac{\partial A_{j\alpha}}{\partial r_l}+v_\alpha\frac{\partial A_{\alpha l}}{\partial r_j}+A_{jl}\frac{\partial v_j}{\partial r_k}+A_{ik}\frac{\partial v_i}{\partial r_j}=0 \tag{58}$$

Hence, by reordering and changing some repeated indices, we have

$$-\frac{\partial A_{kl}}{\partial t}+\frac{\partial(A_{k\alpha}v_\alpha)}{\partial r_l}+\frac{\partial(A_{\alpha l}v_\alpha)}{\partial r_k}=0 \tag{59}$$

which can be written in the form

$$\frac{\partial\mathbf{A}_2}{\partial t}-2\,\nabla_{\mathbf{r}}\star(\mathbf{A}_2\cdot\mathbf{v})=(\mathbf{0})^2 \tag{60}$$

A.4. Property

If we take the colon product of $\mathbf{M}_2$ with Eq. 60, by Eq. 38 and by changing some repeated indices, we can write

$$M_{ij}\frac{\partial A_{ij}}{\partial t} - M_{ij}\frac{\partial A_{j\alpha}}{\partial r_i}v_\alpha - M_{ij}A_{j\alpha}\frac{\partial v_\alpha}{\partial r_i} - M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j}v_\alpha - M_{ij}A_{i\alpha}\frac{\partial v_\alpha}{\partial r_j} =$$

$$= M_{ij}\frac{\partial A_{ij}}{\partial t} - M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j}v_\alpha - \delta_{i\alpha}\frac{\partial v_\alpha}{\partial r_i} - M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j}v_\alpha - \delta_{j\alpha}\frac{\partial v_\alpha}{\partial r_j} = \tag{61}$$

$$= M_{ij}\frac{\partial A_{ij}}{\partial t} - 2M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j}v_\alpha - 2\frac{\partial v_\alpha}{\partial r_\alpha} = 0$$

Now we apply Eq. 48 and Eq. 49 with $\xi = t$ to the first summation term of the last equation, and we also apply the property expressed by Eq. 50 to the second summation term, so that we obtain

$$\frac{\partial \ln|A|}{\partial t} + \frac{\partial \ln|A|}{\partial r_\alpha}v_\alpha - 2\frac{\partial v_\alpha}{\partial r_\alpha} = 0 \tag{62}$$

Thus, the foregoing relation gives account of the divergence of the centroid velocity,

$$\nabla_{\mathbf{r}} \cdot \mathbf{v} = \frac{\partial \ln|A|^{\frac{1}{2}}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\ln|A|^{\frac{1}{2}} \tag{63}$$

A.5. Equation of order $n = 1$

By substitution of the stellar density N, Eq. 7, into the equation corresponding to $n = 1$, Eq. 14, we write

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\mathbf{v} + \nabla_{\mathbf{r}}\mathcal{U} = -\frac{1}{2}\nabla_{\mathbf{r}}\ln|A|\cdot\mathbf{M}_2 - \frac{1}{2}\nabla_{\mathbf{r}}\sigma\cdot\mathbf{M}_2 + \nabla_{\mathbf{r}}\cdot\mathbf{M}_2 \tag{64}$$

Then, by taking into account Eq. 53, we have

$$\nabla_{\mathbf{r}}\cdot\mathbf{M}_2 - \frac{1}{2}\nabla_{\mathbf{r}}\ln|A|\cdot\mathbf{M}_2 = (\mathbf{0}) \tag{65}$$

Therefore, Eq. 64 reduces to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\mathbf{v} + \nabla_{\mathbf{r}}\mathcal{U} = -\frac{1}{2}\mathbf{A}_2^{-1}\cdot\nabla_{\mathbf{r}}\sigma \tag{66}$$

A.6. Equation of order $n = 0$

Let us write the continuity equation Eq. 13 in the following form

$$\frac{\partial \ln N}{\partial t} + \nabla_{\mathbf{r}}\cdot\mathbf{v} + \mathbf{v}\cdot\nabla_{\mathbf{r}}N = 0 \tag{67}$$

Then, by substitution of N, Eq. 7, and by reordering terms, we can write

$$-\frac{1}{2}\left(\frac{\partial \sigma}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\sigma\right) - \left(\frac{\partial \ln|A|^{\frac{1}{2}}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\ln|A|^{\frac{1}{2}}\right) + \nabla_{\mathbf{r}}\cdot\mathbf{v} = 0 \tag{68}$$

Nevertheless, the relationship we obtained for $\nabla_{\mathbf{r}} \cdot \mathbf{v}$ in Eq. 63 makes also null the above first term, which is independent from the tensor $\mathbf{A}_2$. Hence we have

$$\frac{\partial \sigma}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \sigma = 0 \tag{69}$$

Thus, σ is conserved along the centroid trajectory.

Appendix B

Generalised Schwarzschild distribution expressed as a power series of Schwarzschild distributions

Let $\psi(Q+\sigma)$ be a square-integrable function over the interval $I = (0, +\infty)$ in regard to the variable Q, so that it is denoted as $\psi \in \mathcal{L}^2(I)$. Notice that ψ can not be constant. Let us remember that Q depends on the velocities through the positive definite quadratic form defined in Eq. 5, σ depends only on time and position, and ψ is the velocity distribution. Then ψ may be written as[6]

$$\psi(Q+\sigma) = F\left(\frac{Q+\sigma}{2}\right) e^{-\frac{1}{2}(Q+\sigma)} \tag{70}$$

where F can be expressed as a series of Laguerre polynomials, which are an orthogonal basis of the vector space $\mathcal{L}^2(I)$ with respect to the weight $w(Q) = e^{-\frac{1}{2}(Q+\sigma)}$ on I.
Hence, by defining the variable

$$\tau = \frac{1}{2}(Q+\sigma) \tag{71}$$

we can write

$$F(\tau) = \sum_{k=0}^{\infty} \alpha_k L_k^0(\tau) \tag{72}$$

where each Fourier coefficient α_k, which multiplies the Laguerre polynomial L_k^α with $\alpha = 0$, can be evaluated from the inner product defined as

$$\langle f(\tau), g(\tau)\rangle = \int_0^\infty f(\tau) g(\tau) e^{-\tau} d\tau \tag{73}$$

On the other hand, the foregoing conditions do also allow to compute the central velocity moments, which are given through the integral Eq. 33. Thus, bearing in mind the Eq. 70, the integral takes the form

$$\phi_n(\sigma) = \int_0^\infty Q^{\frac{n}{2}} F\left(\frac{Q+\sigma}{2}\right) e^{-\frac{1}{2}(Q+\sigma)} Q^{\frac{1}{2}} dQ \tag{74}$$

[6]Let us remember that any square-integrable function $f(x) \in \mathcal{L}^2(I)$ admits an expansion as a series of the associated Laguerre functions,

$$f(x) = \sum_{k=0}^{\infty} L_k^\alpha(x) x^\alpha e^{-x}$$

with $\alpha > -1$, where $\{L_k^\alpha(x)\}_{k\in\mathcal{N}}$ is the family of the associated Laguerre polynomials, which is an orthogonal basis of the space $\mathcal{L}^2(I)$ with respect to the weight $x^\alpha e^{-x}$ (e.g. Abramowitz & Stegun 1965).

Notice that such an integral is convergent because, after the change of variable given in Eq. 71, F can be expressed through the associated Laguerre polynomials, now with $\alpha = \frac{1}{2}$.
The above polynomial form of F can be formally useful in the case we want to estimate it from all the available central moments. Nevertheless, to our current purpose, we need to write F depending on the new variable

$$\eta = e^{-\tau} = e^{-\frac{1}{2}(Q+\sigma)} \tag{75}$$

Then, for a fixed σ it is possible to establish an isomorphism between the respective domains of Q and η, namely $I = (0, +\infty)$ and $J = (0, e^{-\frac{1}{2}\sigma})$, so that with the notation

$$\tilde{F}(\eta) = F(\tau(\eta)) \tag{76}$$

the function $\tilde{F}(\eta)$ can be expressed as depending on the new variable from a basis of orthogonal polynomials $\{P_k(\eta)\}_{k\in\mathcal{N}}$ over J, according to an inner product which is equivalent to the previous one defined on I from Eq. 73. That is,

$$\begin{aligned} \langle f(\tau), g(\tau)\rangle_I \equiv \int_0^\infty f(\tau)g(\tau)e^{-\tau}d\tau = \int_0^{e^{-\frac{1}{2}\sigma}} f(\tau(\eta))g(\tau(\eta))e^{-\tau(\eta)} \left|\frac{d\eta}{d\tau}\right|^{-1} d\eta = \\ = \int_0^{e^{-\frac{1}{2}\sigma}} \tilde{f}(\eta)\tilde{g}(\eta)d\eta \equiv \langle \tilde{f}(\eta), \tilde{g}(\eta)\rangle_J \end{aligned} \tag{77}$$

Thus, we can write $\tilde{F}(\eta)$ as a series of polynomials $\{P_k(\eta)\}_{k\in\mathcal{N}}$ or, by reorganising terms, as power series of η in the following form

$$\tilde{F}(\eta) = \sum_{k=0}^{\infty} \beta_k P_k(\eta) = \sum_{k=0}^{\infty} \gamma_k \eta^k \tag{78}$$

Hence, according to Eq. 76, by substitution of Eq. 78 into Eq. 70, and by writing it in terms of the variable Q, we finally have

$$\psi(Q+\sigma) = e^{-\frac{1}{2}(Q+\sigma)} \sum_{k=0}^{\infty} \gamma_k e^{-\frac{1}{2}(Q+\sigma)\,k} = \sum_{k=1}^{\infty} \gamma_{k-1} e^{-\frac{1}{2}(Q+\sigma)\,k} \tag{79}$$

Thus, any arbitrary quadratic function $\psi(Q+\sigma)$ can be expressed as a convergent series of the Gaussian functions $e^{-\frac{1}{2}(Q+\sigma)\,k}$, with $k \geq 1$, although they are not an orthogonal system.

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